Application of Game Theory in Inventory Management

Rodrigo Tranamil-Vidal

Universidad de Chile, Santiago de Chile, Chile
Rodrigo.tranamil@ug.udechile.cl

Abstract. Game theory has been successfully applied in supply chain management problems due to its capacity of modeling situations where companies have to make strategic decisions about their production planning, inventory control and distribution systems. In particular, this article presents the application of game theory in inventory management. First, we present the basics concepts of non-cooperative and cooperative game theory. Then, we present inventory models by means of game theory. For each model, we provide its configuration, the solution concept implemented, and the existence and uniqueness of the equilibrium used.

Keywords: inventory control, non-cooperative game theory, Nash equilibrium, cooperative game theory, the core, the Shapley value.

1 Introduction

This work presents briefly some basic concepts of game theory and shows its application in inventory management by using modifications of the classic news vendor model. In the traditional news vendor model, the inventory problem is modeled as a single decision maker problem (for example, a retailer that wants to define an optimal order quantity for a particular product) where the decisions are made without considering other competitors (for example, other retailers o firms that provide the same o similar product in the marketplace), and if there are different types of products, they cannot be substituted for each other. The problem with these assumptions is that in many real situations this is not true. Now, the question that arises here is: can we construct an inventory model that is able to describe such situations? The answer is YES. In particular, we can use Game Theory.

Game theory provides mathematical tools to model strategic interactions in which there are several players (decision makers) that want to maximize their benefits by playing a certain strategy that considers the strategies of the other players[1]. The games can be divided into two branches:

[1] In inventory problems the strategy is the order quantity

- **Non-Cooperative Games**: players are unable to make binding commitments regarding which strategy they will choose. Therefore, decisions are made independently.
- **Cooperative Games**: groups of players can form coalitions. So they are able to make binding commitments with side-payments.

The following sections show how each of these game theoretic models can be applied in inventory problems with several decision makers and substitutable products. The definitions used in this work came from Osborne & Rubinstein (1994).

2 Non-cooperative Games

This section presents the basic theory behind the non-cooperative model. We start giving the formal definition of strategic games. We then continue with the definition of Nash equilibrium as solution concept. We focus on the Nash equilibrium because is the solution concept most used in the inventory management literature. We end this section showing the game setup of the non-cooperative news-vendor model and its evolution in the associated literature.

2.1 Strategic Games

In a strategic game, a decision-maker chooses his action once and simultaneously. Formally,

Definition 2.1 A strategic game $G = \langle N, A_i, u_i \rangle$ consists of

- $N = \{1, .. n\}$ set of n players
- $A_i = \{a_1^i, \dots, a_n^i\}$ set of actions or strategies available to player $i \in N$
- $A = \times_{i \in N} A_i$ set of action of strategy of the game G
- $u_i : A \rightarrow \mathbb{R}$ the utility function for player $i \in N$

Two important observations from the definition 2.1 are

(a) If player i knows A_j, $\forall j \in N \backslash \{i\}$, then the game has perfect information. Otherwise, it is a game of incomplete information.

(b) If no two players choose their actions simultaneously, then the game is dynamic. Otherwise, it is a static game.

The outcome of the strategic game G is the profile $a = (a^i, \dots, a^n) \in A$ where $a^i \in A_i$, $i \in N$. This outcome is called pure strategy if the players are not mixing their actions. For example, in the case where A_i is known for everyone, the player i may want to flip a coin to mix his strategy in order to confuse his competitors.

In this article we focus on pure strategies of static games of complete information. Now, how can we find such strategies? The following subsection is devoted to answer the previous question.

2.2 Solution concepts for Strategic Games

There are several solution concepts to find a pure strategy of the strategic game G. One of them is the Nash equilibrium which is widely used in non-cooperative games. The idea is that the equilibrium occurs if no player wants to deviate because it would lead to lower payoff. The formal definition is as follows[2]:

Definition 2.2 A Nash equilibrium of a strategy game $G = \langle N, A_i, u_i \rangle$ is a profile $a^* \in A$ such that for every player $i \in N$, $u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \forall a_i \in A_i$.

Other way to define the Nash equilibrium is by using the notion of the best response,

Definition 2.3 The best response (function) to a_{-i} is the strategy that maximizes player i's payoff. Mathematically, $B_i(a_{-i}) = \operatorname{argmax}_{a_i} u_i(a_i, a_{-i})$. Therefore, the profile $a^* \in A$ is a Nash equilibrium of the game G if $a_i^* \in B_i(a_{-i}), \forall i \in N$.

Some observations:

(a) From definition 2.2, Nash equilibrium is a solution to a system of n equations which implies that the equilibrium may not exist. This is a problem because our primary goal is to find the outcomes of the game G.

(b) From definition 2.3, Nash equilibrium is a solution of a maximization problem which implies that the equilibrium may not be unique. The problem with multiple equilibriums is that we do not know which one will prevail in practice.

(c) Note that the definition 2.3 is more useful than 2.2 when we need to compute the Nash equilibria.

The following subsection shows some methodologies to determinate the existence and uniqueness of the Nash equilibrium.

2.3 Properties of the Nash Equilibrium

Existence: There are some simple ways to show that at least one Nash equilibria exists. Since a Nash equilibrium is a fixed point of the best response mapping, then the fixed point theorem can be used. The theorem is the following,

[2] Consider the player $i \in N = \{1, \ldots, n\}$. The strategies of the other players a_{-i} is given by the vector $a_{-i} = \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$

Theorem 2.3 If for each $i \in N$, A_i is compact and convex, and u_i is continuous and quasi-concave, then there exists at least one Nash equilibrium in the game G.

Note that sometimes theorem 2.3 is difficult to use. In some situations it is much easier just to verify the concavity of the players payoffs u_i, but we need continuous best response functions to do this.

Uniqueness and Multiple equilibriums: It is important to know if a Nash equilibrium exists. It is also important to know if only one equilibrium exists because this implies that the actions in equilibrium will be observed in practice. However, if we have multiple equilibria, then we cannot establish which one will prevail for sure. Some ways to show uniqueness are: algebraic argument (looks the optimality conditions), contraction mapping argument (shows that the best response mapping is a contraction), univalent mapping argument (verify that the best response function is one-to-one).

We have defined the basic theory of non-cooperative games so far. Now, we will devote the following subsection to describe the competitive new-vendor model as an extension of the classic new-vendor model. In this extension, the products are substitutable and hence the players (retailers) compete for the best order quantity to satisfy the demand.

2.4 Application in Inventory Management

The Competitive News-vendor Model.

Consider the classic news-vendor model. In addition, we have substitution between different products sold by different retailers. Also, if a customer does not find a product in the first firm, then she may travel to another firm in order to satisfy her demand. Hence, each retailers profit depends not only on her own order quantity but also on her competitors' orders. We use the game setup given by Cachon & Netessine (2004),

- For simplicity, consider two retailers i and j facing stochastic demands w_i and w_j with c.d.f $F(w_i)$ and $F(w_j)$ correspondingly.
- Firms must choose an inventory level $x_i \in [0, \infty)$ at cost c_i per unit for the perishable product that they provide in the marketplace.
- Suppose shortage cost p_i, salvage value s_i.
- This model suppose that the excess demand at firm i is satisfies by firm j.
- The actual demand of firm j is $R_j = w_j + \beta_i(w_i - x_i)^+$ where $\beta_i \in [0, 1]$ is the substitution rate.

- Suppose that the profit of the firm i is given by the function $\pi_i(x_i, x_j)$.

Given the previous definitions, the expected profit for the firm j is defined by

$$E[\pi_j(x_i, x_j)] = r_j \, E \, min\{x_j \, , R_j\} + s_j \, E(x_j - R_j)^+ - c_j x_j - p_j \, E(R_j - x_j)^+$$

Note that player's i best response (function) is the strategy x_i^* that maximizes the player's i payoff. Following definition 2.3, the best response function is

$$B_i(x_j) = \text{argmax}_{x_i} E\left(\pi_j(x_i, x_j)\right), \forall \, x_j$$

Hence, the pair of inventory levels (x_i^*, x_j^*) is a Nash equilibrium if

$$x_i^* = \text{argmax}_{x_i} E\left(\pi_j(x_i, x_j)\right)$$

$$x_j^* = \text{argmax}_{x_j} E\left(\pi_j(x_i, x_j)\right)$$

Now, from the classic news-vendor model we know that

$$x^* = \text{argmax}_x \, E(\pi) = F_W^{-1}\left(\frac{r - c + s}{r + p - s}\right)$$

Therefore, we can establish that the Nash equilibrium of the competitive news-vendor model defined in this work is given by

$$x_i^*(x_j^*) = F_{R_j^*}^{-1}\left(\frac{r_i - c_i + s_i}{r_i + p_i - s_i}\right)$$

$$x_j^*(x_i^*) = F_{R_i^*}^{-1}\left(\frac{r_j - c_j + s_j}{r_j + p_j - s_j}\right)$$

What about existence and uniqueness in this game?

It is easy to check that the news-vendors objective function is quasi-concave with respect to the order quantity (look second derivative). Hence, there exists equilibrium in the competitive news-vendor model.

Now, the best response mapping is a contraction (see Chinchuluun et al. (2008) for more details) because the slopes of the best response functions satisfy $\left|\frac{\partial x_i^*(x_j)}{\partial x_j}\right| < 1$. Hence, the equilibrium in the competitive news-vendor model is unique.

Extensions.

A brief review of the literature is the following:

1. **Competition in horizontal channels**
   (a) Nti (1987): inventory model with n competitive firms.
   (b) Parlar (1988): inventory model with substitutable products.
   (c) Lippman & McCardle (1997): competitive newsboy model in both oligopoly and duopoly contexts.
   (d) Mahajan & van Ryzin (1997): n-firm inventory competition with dynamic demand.
   (e) R. Anupindi & Zemel (2001): decentralized inventory systems with n retailers.
2. **Competition in vertical channels**
   (a) Cachon & Netessine (1999): two echelon competitive inventory model with one-supplier, one-retailer.
   (b) Cachon & Zipkin (1999): two-stage supply chain with fixed transportation times.
   (c) Cachon (2001): extended the above models to n retailers.
   (d) H. Wang & Efstathiou (2001): one-suplier and n-retailers inventory model.

In the following section, we will present the theory associated with the cooperative game model, and then we will show the cooperative new-vendor model.

3 Cooperative Game Theory

This section presents the basic theory behind the cooperative model. We start giving the formal definition of coalitional games. There are two main issues in this kind of models: formation of coalitions, and allocation of the benefits among the members of the coalitions. The Core and the Shapley value are solution concepts that help to understand and solve the issues mentioned before. We end this section showing the game setup of the cooperative news-vendor model and its review of literature.

3.1 Coalitional Games

In the cooperative model communication between players is allowed. Hence, the players can form coalitions because it may result in higher benefits. The definition is

Definition 3.1 A coalitional game $G = \langle N, v \rangle$, consists of

- $N = \{1, \ldots, n\}$ set of players.
- $v : 2^N \to \mathbb{R}$ the characteristic function of coalition $S \subseteq N$.

From definition 3.1, the following questions can be pointed out

(a) How can we know if the coalition S is stable?

(b) Given that $v(S)$ is the worth of the coalition S, how can we fairly allocate $v(S)$ among the players in S?

3.2 Solution concepts for Coalitional Games

The Core.

The Core is a solution concept which is used to verify if a coalition S is stable. The idea is that a coalition is stable if no deviation is profitable. To give a formal definition of the Core, we first shall define allocation. Let x_i be the feasible payoff of player i. The allocation vector of the game defined in 3.1 is $X = (x_1, \ldots, x_N)$. An allocation define how much each player will receive if they play as member of coalition S. Now we are able to establish coalitional stability by means of the allocation vector.

Definition 3.2 The Core of the cooperative game G is the set

$$C(v) = \left\{ X \in \mathbb{R}^N : \sum_{i}^{N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \right\}$$

Suppose that the game G has the the allocation vector $X = \left(x_1, \ldots, x_{|N|}\right)$. Then we can say that the coalitions in game G are stable if $X \in C(v)$. An issue with definition 3.2 is that it may be empty. In such case, we cannot establish stability for the game G. There several ways to check if the core is nonempty, two of them are the following results

- $C(v)$ is nonempty $\Leftrightarrow$ $v(S)$ is balanced.
- If $v(S)$ is convex, then $C(v)$ is nonempty.

The Shapley Value.

The Shapley Value is a solution concept to compute fair allocations of the game G. It is defined axiomatically as follows,

Axiom 1: Symmetries in v, the value of the players should not change due to permutations of players.

Axiom 2: Irrelevance of a dummy player, only players generating added value should share the benefits.

Axiom 3: The sum of two games, $\pi(v_1 + v_2, N) = \pi(v_1, N) + \pi(v_2, N)$.

The unique allocation that satisfies the previous axioms is the Shapley value $\phi_i(v)$ which is given by the following expression

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(|N| - 1 - |S|)!}{|N|!} \Big(v\big(S \cup \{i\} - v(S)\big) \Big)$$

Note that the Shapley Value selects one fair payoff vector $X = (\phi_1(v), \dots, \phi_n(v))$ for the game G defined in 3.1.

Now, we will devote the following subsection to describe the cooperative new-vendor model as an extension of the classic new-vendor model.

3.3 Application in Inventory Management

Cooperative News-vendor Model.

The basic assumption in this model is that retailers may increase their profit if they decide to cooperate. The basic cooperation rules that could appear in this setting are: switch excess inventory to anyone who has excess demand so that the latter can save in lost sales cost, or give a joint order and use this quantity to satisfy the total demand they are facing so that they can save in ordering cost.

We use the game setup given by M. Slikker & Wouters (2001),

- In this model we have N retailers facing the stochastic demand w_i , $i = \{1, \dots, |N|\}$.
- Suppose wholesale price c_i and selling price r_i.
- Also suppose a transshipping cost t_{ij} per unit from retailer i to retailer j.
- Let X^Sbe the set of possible order quantity vector of coalition $S \subseteq N$.

- The order quantity vector is given by $x_S \in X^S$.
- Let A_{ij}^S be the amount of products that are transshipped from retailer i to j.

By using the previous definitions, we can write the profit of the coalition S as follows

$$\pi^S(x^S) = \sum_{j \in S} r_l \min\left\{\sum_{i \in S} A_{ij}^S, x_l^S\right\} - \sum_{i \in S}\sum_{j \in S} A_{ij}^S t_{ij} - \sum_{i \in S} c_i x_i^S$$

Now, we can establish the characteristic function v of this game. Given the realized demand $w = (w_1, \dots, w_{|N|})$, the worth of the coalition S is the maximum value of $\pi^S(x^S)$. That is,

$$v(S) = \max_{x \in X^S} E_w[\pi^S(x)/ w^S], \forall S \subseteq N$$

M. Slikker & Wouters proved that

1. There exists coalitions $S \subseteq N$ for this cooperative game G.
2. There exists x^S that maximizes $E_w[\pi^S(x)/ w^S]$
3. The cooperative news-vendor game has a nonempty core.

Extensions.

A brief review of the literature is the following. Note that there are a few papers talking about the application of cooperative game theory in inventory management.

1. Competition in horizontal channels

(a) Gerchak & Gupta (1991): joint inventory control among n retailers and one supplier

(b) Hartman & Dror (1996): centralized and continuous-review inventory system

(c) N. Rudi & Pyke (2001): two-location inventory with transshipment.

2. Competition in vertical channels

(a) Raghunathan (2003): one manufacturer, n-retailer supply chain with correlated demand at retailers.

4 Final Comments

In this work we briefly have discussed some basic concepts of both non-cooperative and cooperative game theory. Also, we reviewed applications of game theory in some extensions of the classic news-vendor model. It is important to point out that there is more equilibrium analysis in supply chain management. For example,

1. **Production and Pricing Competition**: Vertical competition between a manufacturer and a retailer, or horizontal competition between manufacturers or retailers.

2. **Capacity decisions**: capacity-constrained systems where a supplier provides products to several retailers.

3. **Joint inventory decisions**: firms order jointly and decide how to allocate the inventory savings.

4. **Channel coordination**: allocate inventory risk between suppliers and retailers via different types of contracts (buy-back, quantity discount, etc).

It is also important to comment on directions for future research

1. **Inventory Games**: More papers focus on decentralized channel. It could be a good direction to research on centralized inventory.

2. **Multi-stage inventory models**: Two examples are,

   (a) Retailers make order decision at the first stage. At the second stage, they decide how much inventory to transship among locations to better match demand.

   (b) Inventory procurement. Then, decide how much inventory to share with others. And then, decide how much inventory to transship.

3. **Information asymmetry in supply chain contracts**: Supplier offers a menu of contracts. The idea is that the manufacturer chooses the contract that the supplier wants to implement.

5 References

Cachon, G. (2001). Stock wars: Inventory competition in a two-echelon supply chain with multiple retailers. Operations Research, 49(5), 658–674.

Cachon, G., & Netessine, S. (1999). Game theory in supply chain analysis. In R. G. S. Tayur & M. Magazin (Eds.), Quantitative models for supply chain management. Boston: Kluwer Academic Publisher.

Cachon, G., & Netessine, S. (2004). Game theory in supply chain analysis. In D. Simchi-Levi, S.Wu, & Z. Shen (Eds.), Handbook of quantitative supply chain analysis: Modeling in the e-business era. Boston: Kluwer Academic Publisher.

Cachon, G., & Zipkin, P. (1999). Competitive and cooperative inventory policies in a two-stage supply chain. Management Sciences, 45(7), 936–953.

Chinchuluun, A., Karakitsiou, A., & Mavrommati, A. (2008). Game theory models and their applications in inventory management and supply chain. In A. Chinchuluun, P. Pardalos, A. Migdalas, & L. Pitsoulis (Eds.), Pareto optimality, game theory and equilibria. New York: Springer.

Gerchak, Y., & Gupta, D. (1991). On apportioning costs to customers in centralized continuous review inventory systems. Journal of Operations Management , 10(4), 546–5519.

Hartman, B., & Dror, M. (1996). Cost allocation in continuous-review inventory models. Naval Research Logistics, 10(4), 546–5519.

H. Wang, M. G., & Efstathiou, J. (2001). A game-theoretical cooperative mechanism design for a two-echelon decentralized supply chain. European Journal of Operational Research, 157(2), 372–388.

Lippman, S., & McCardle, K. (1997). The competitive newsboy. Operations Research, 45 , 54–65.

Mahajan, S., & van Ryzin, G. (1997). Inventory competition under dynamic consumer choice. Operations Research, 49, 646–657.

Mingming, L., & Mahmut, P. (2005). Game theoretic applications in supply chain management: A review. Information Systems and Operational Research (INFOR), 43, 187–220.

M. Slikker, J. F., & Wouters, M. (2001). Cooperation between multiple news-vendors with transshipments. European Journal of Operational Research, 167, 370-380.

N. Rudi, S. K., & Pyke, D. (2001). A two-location inventory model with transshipment and local decision making. Management Science, 47, 1668–1680.

Nti, K. (1987). Competitive procurement under demand uncertainty. Management Sciences, 33 , 1489–1505.

Osborne, M., & Rubinstein, A. (1994). A course in game theory. Cambridge, Massachusetts: MIT Press.

Parlar, M. (1988). Game theoretic analysis of the substitutable product inventory problem with random demands. Naval Research Logistics, 35, 397–4090.

Raghunathan, S. (2003). Impact of demand correlation on the value of incentive for information sharing in a supply chain. European Journal of Operational Research, 146(3), 634–649.

R. Anupindi, Y. B., & Zemel, E. (2001). A general framework for the study of decentralized distribution system. Manufacturing and Service Operations Management , 3(4), 349–368.